**Differential Equations - Unit 1 Notes**

Definitions and Review

**Definition** A first-order differential equation is an equation for an unknown function in terms of its derivative. The standard form for such an equation is

\[
\frac{dy}{dt} = f(t, y),
\]

meaning the RHS (right-hand side) can be a function of both \( t \) and/or \( y \).

**Definition** A solution of the differential equation is a function of the independent variable that, when substituted into the equation as the dependent variable, satisfies the equation for all values of the independent variable, i.e., if \( y(t) \) satisfies \( \frac{dy}{dt} = y'(t) = f(t, y(t)) \).

**Example 1** Consider \( \frac{dy}{dt} = y \). Note that \( y_1(t) = 3e^t \) is a solution, but \( y_2(t) = \sin t \) is not a solution because

\[
\frac{d}{dt}(3e^t) = 3e^t = y
\]

but

\[
\frac{d}{dt}(\sin t) = \cos t \neq y.
\]

**Example 2** Consider \( \frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t} \). Check whether the following functions are solutions:

- \( y_1(t) = 1 + t \)
- \( y_2(t) = 1 + 2t \)
- \( y_3(t) = 1 \)

**Definition** A differential equation along with an initial condition is called an initial value problem. We seek a solution \( y(t) \) of the given equation that assumes a given value \( y_i \) at a particular time \( t_i \).

**Example 3** The initial value problem (IVP)

\[
\frac{dy}{dt} = t^3 - 2\sin t, \quad y(0) = 3
\]

can be solved using the techniques we learned in our calculus class. Note that the RHS (right-hand side) of the differential equation depends only on \( t \), so this is an anti-differentiation problem in which we (1) antidifferentiate and add a constant of integration, (2) use the initial condition to solve for the constant of integration, and (3) rewrite the result of the antidifferentiation with the newfound value of the constant of integration. Carry out these steps for this particular problem.

**Solution** METHOD 1: The straightforward antidifferentiation technique learned in our calculus class yields

\[
y(t) = \frac{t^4}{4} + 2\cos t + C, \quad (1)
\]

and with \( y(0) = 3 \), we have

\[
3 = 2 + C, \quad \text{or} \quad C = 1,
\]

and finally,

\[
y(t) = \frac{t^4}{4} + 2\cos t + 1. \quad (2)
\]

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Solution METHOD 2: A technique we did not use very often in calculus class involves a more obvious use of the FTC (Fundamental Theorem of Calculus). Recall that \( \int_a^b f(t) \, dt = F(b) - F(a) \). Then

\[
\int_0^t y'(x) \, dx = y(t) - y(0) = y(t) - 3.
\]

Equivalently,

\[
y(t) = \int_0^t y'(x) \, dx + 3
\]

\[
= \int_0^t (x^3 - 2 \sin x) \, dx + 3
\]

\[
= \left. \frac{x^4}{4} + 2 \cos x \right|_0^t + 3
\]

\[
= \left( \frac{t^4}{4} + 2 \cos t \right) - (0 + 2) + 3
\]

\[
= \frac{t^4}{4} + 2 \cos t + 1.
\]

In either case, the solution containing the yet-to-be-determined constant of integration \( C \) is called the general solution (because we can use it to solve any initial value problem) and the solution in which we have used the initial condition (IC) to solve for the constant of integration \( C \) is called the particular solution (because it is particular to that initial condition).

Separable Differential Equations

We know how to check solutions to differential equations (this is always possible, provided we know how to differentiate the functions involved). We also know how to find solutions to initial value problems in which the resulting integrand involves only the independent variable \( t \) and is actually integrable. But how do we find solutions in general? Quite often, typical first-order equations are given in the form

\[
\frac{dy}{dt} = f(t, y)
\]

in which the RHS involves functions of both \( t \) and \( y \). (We have also seen at least one case where the RHS only involves \( t \). We will also see cases in which it only involves \( y \).) Recall the following from our work in calculus.

**Definition** A differential equation is separable if it can be written as a product of a function of the dependent variable and a function of the independent variable, i.e., it can be written in the form

\[
\frac{dy}{dt} = g(t) \, h(y).
\]

If this form can be achieved, the differential equation can be solved by rewriting the equation as

\[
\frac{dy}{h(y)} = g(t) \, dt
\]

and integrating

\[
\int \frac{dy}{h(y)} = \int g(t) \, dt,
\]

provided \( 1/h(y) \) and \( g(t) \) are integrable functions of \( y \) and \( t \) respectively.

**Example 4** The equation \( \frac{dy}{dt} = yt \) is separable as \( \frac{du}{y} = t \, dt \).

**Example 5** The equation \( \frac{dy}{dt} = y + t \) is not separable.
Example 6 The equation \( \frac{dy}{dt} = \frac{t+1}{ty+1} \) is separable as \((y + 1) \, dy = \frac{t+1}{t} \, dt\) (confirm this!)

Example 7 The equation \( \frac{dy}{dt} = g(t) \) is always separable (but not necessarily solvable) with \(1 \cdot dy = g(t) \, dt\) (this was a typical problem in our calculus classes).

Example 8 The equation \( \frac{dy}{dt} = h(y) \) is always separable (but not necessarily solvable) with \(\frac{1}{h(y)} \, dy = \, dt\). This equation involves only the dependent variable and is called an autonomous differential equation. Many of the first-order differential equations that arise in applications are autonomous, like the population models we studied in our previous investigation, such as \( \frac{dP}{dt} = kP \left( 1 - \frac{P}{M} \right) \).

The following examples show some of the techniques in solving separable differential equations as well as some of the pitfalls one might encounter.

Example 9 (STANDARD) To solve

\[ \frac{dy}{dt} = \frac{t}{y^2}, \]

we use "informal" algebra to rewrite it as the integral equation

\[ \int y^2 \, dy = \int t \, dt \]

and antidifferentiate both sides to obtain

\[ \frac{y^3}{3} = \frac{t^2}{2} + c. \]

This equation can be solved for \( y(t) \) explicitly as

\[ y(t) = \left( \frac{3t^2}{2} + 3c \right)^{1/3} = \left( \frac{3t^2}{2} + c_1 \right)^{1/3}. \]

(Note the placement of the "arbitrary" constant of integration and how the algebraic steps operate on it as well.)

Example 10 (MISSING SOLUTIONS) Consider

\[ \frac{dy}{dt} = y^2, \]

an autonomous and hence separable equation. Rewriting and integrating yields

\[ \int \frac{dy}{y^2} = \int dt \]
\[ -\frac{1}{y} = t + c \]
\[ y(t) = -\frac{1}{t + c}. \]

However, the equation presented in (3) is not the general solution (i.e., it cannot solve ALL initial value problems!) What if \( y(0) = 0 \)? There is no value of \( c \) for which

\[ y(0) = -\frac{1}{0 + c} = 0. \]

Here is a case in which we are unable to obtain all possible solutions with the technique of separation of variables. Why? Note that the particular solution that contains the initial condition \( y(0) = 0 \) is the constant equilibrium solution \( y(t) = 0 \) (because \( dy/dt = 0 \) and \( y^2 = 0 \), which satisfies the original equation). So in its totality, the general solution, the solution that can solve all initial value problems, contains all solutions of the form given in (3) along with \( y(t) = 0 \). One should always look for these "obvious" solutions before performing extensive algebraic manipulations to determine the other solutions.
Example 11 (IMPLICIT) The equation

$$\frac{dy}{dt} = \frac{y}{1+y^2}$$

is autonomous, hence separable. We obtain

$$\int \frac{1+y^2}{y} dy = \int dt$$

$$\ln |y| + \frac{y^2}{2} = t + c.$$  

However, this equation is not solvable explicitly for $y(t)$ alone, so we are left with this implicit solution, which is often the best we can do. Again, note that the RHS of the original differential equation vanishes (becomes zero) if $y = 0$, so the constant function $y(t) = 0$ again serves as an equilibrium solution that cannot be obtained from the solution derived via separation of variables.

Example 12 (DOH!) The equation

$$\frac{dy}{dt} = \sec (y^2)$$

is autonomous, hence separable. However, the resulting integral

$$\int \cos y^2 dy = \int dt$$

is impossible to evaluate (in fact, its evaluation leads to the definition of an entire class of integrals called the "Fresnel integrals"). We cannot always depend on finding analytic solutions even when we can separate variables!

Applications of Separable Differential Equations

A Savings Model

Example 13 We deposit $5000 in a savings account with interest accruing at the rate of 5% compounded continuously. If $A(t)$ is the amount of money in the account at time $t$, then the rate at which this amount changes is given by the familiar differential equation

$$\frac{dA}{dt} = 0.05A,$$

having general solution $A(t) = ce^{0.05t}$ and particular solution $A(t) = 5000e^{0.05t}$. If rates remain constant over the next 10 years, our account will have $A(10) = 5000e^{0.5} \approx$ $8243.61 in our account. After this initial 10 years, we decide to withdraw $1000 each year. Will we ever go broke?

Solution Our previous model works fine for $0 \leq t \leq 10$. For $t > 10$, we’d have a new rate of change, namely

$$\frac{dA}{dt} = 0.05A - 1000,$$

and our overall model is given by

$$\frac{dA}{dt} = \begin{cases} 0.05A, & \text{for } 0 \leq t \leq 10; \\ 0.05A - 1000, & \text{for } t > 10, \end{cases}$$

which of course is a piecewise definition whose graph is shown below. Given that $A(10) \approx$ $8243.61, we proceed to solve the second part of this equation. This is a separable equation, giving

$$\int \frac{dA}{0.05A - 1000} = \int dt$$
and with \( u = 0.05A - 1000 \) so that \( du = 0.05dA \) (and \( dA = 20du \)) we have

\[
\int \frac{20du}{u} = \int dt
\]

\[
20 \ln |u| = t + c
\]

\[
20 \ln |0.05A - 1000| = t + c
\]

for some \( c \). We can deal with the absolute value at this point. The "initial" condition for this part of the differential equation is actually the value of \( A \) at \( t = 10 \), or \( A(10) \approx 8243.61 \). This means that at \( t = 10 \),

\[
\frac{dA}{dt} = 0.05A - 1000 \approx -587.8195 < 0,
\]

and so \( A \) is decreasing. Note that \( A \) would increase when \( A'(t) > 0 \), or when \( A > 20000 \). However, since \( A \approx 8243.61 \) and is currently decreasing, it will never reach 20000. This tells us that \( 0.05A - 1000 \) is negative and will remain negative for all \( t > 10 \), and so \( |0.05A - 1000| = -(0.05A - 1000) = (1000 - 0.05A) \). This leads us to

\[
20 \ln |0.05A - 1000| = t + c
\]

\[
20 \ln (1000 - 0.05A) = t + c
\]

\[
1000 - 0.05A = e^{0.05(t+c)}
\]

\[
1000 - 0.05A = e^{0.05c}e^{0.05t}
\]

\[
1000 - 0.05A = c_1e^{0.05t},
\]

where \( c_1 = e^{0.05c} \). Solving for \( A \), we find

\[
A = \frac{1000 - c_1e^{0.05t}}{0.05} = 20000 - 20c_1e^{0.05t} = 20000 - c_2e^{0.05t},
\]

where \( c_2 = 20c_1 \). Finally, by the initial condition, we have

\[
A \approx 8243.61 = 20000 - c_2e^{0.05(10)}
\]

\[
\Rightarrow c_2 = -\frac{8243.61 - 20000}{e^{0.5}} \approx 7130.6110,
\]

and so

\[
A(t) = 20000 - 7130.6110e^{0.05t}.
\]

Here is a graph of the original differential equation along with the corresponding solution:

Note the discontinuous derivative, but the continuous solution (because we used the value of \( A(10) \approx 8243.61 \) from the first equation as the initial value for the second equation).
So when will our account be depleted? We need to solve \( A(t) = 0 \) for \( t \):

\[
20000 - 7130.6110e^{0.05t} = 0
\]

\[
t = \frac{\ln \left( \frac{20000}{7130.6110} \right)}{0.05} = 20.627 \text{ years.}
\]

By the way, we could have solved for \( c \) earlier. Starting with \( 20 \ln |0.05 (8243/61) - 100| = 10 + c \) yields \( c = 117.5284 \). Note that the sequence of changes of \( c \to c_1 \to c_2 \) produces the same \( c_2 \) as derived above: \( c_1 = e^{0.05c} = 356.5306 \) and \( c_2 = 20c_1 = 7130.61 \).

### A Mixing Problem

There are plenty of scenarios in which we might mix two or more substances together at various rates, and these problems are collectively known as "mixing problems" (think pollutants in a lake, chemical mixtures, paint mixtures, diffusion of particles in the air, etc.). For example, suppose we have a large vat of sugar water for the making of soft drinks and:

- The vat contains 100 gallons of liquid, and the amount flowing into the vat is the same as the amount flowing out, so this vat always contains 100 gallons.
- The vat is uniformly mixed so the concentration of sugar is the same throughout.
- Sugar water containing 5 tbsp. of sugar per gallon enters the vat through pipe A at a rate of 2 gal/min.
- Sugar water containing 10 tbsp. of sugar per gallon enters the vat through pipe B at the rate of 1 gal/min.
- Sugar water leaves the vat through pipe C at the rate of 3 gal/min.

The independent variable in this case is time \( t \). We could model in terms of the amount of sugar \( S(t) \) or in terms of the concentration \( C(t) \). We’ll use \( S(t) \). The rate of change in the amount of sugar in the vat is the difference between the amount of sugar being added and the amount of sugar being removed. The model (with units) is

\[
\frac{dS}{dt} \text{ tbsp/min} = \frac{2 \text{ gal/min} \cdot 5 \text{ tbsp/gal}}{\text{input from pipe A (tbsp/min)}} + \frac{1 \text{ gal/min} \cdot 10 \text{ tbsp/gal}}{\text{input from pipe B (tbsp/min)}} - \frac{3 \text{ gal/min}}{\text{output rate}} \cdot \frac{S(t) \text{ tbsp}}{\text{total concentration in the vat}} - \frac{3 \text{ gal/min}}{\text{output through pipe C (tbsp/min)}}
\]

Again, this is a separable equation

\[
\frac{dS}{dt} = 20 - \frac{3S}{100} = \frac{2000 - 3S}{100},
\]

whereby separating and integrating yields

\[
\int \frac{dS}{2000 - 3S} = \int \frac{dt}{100}
\]

\[
\frac{\ln |2000 - 3S|}{-3} = \frac{t}{100} + c
\]

\[
\ln |2000 - 3S| = -3t + 3c
\]

\[
\ln |2000 - 3S| = -0.03t + c_1,
\]

where \( c_1 = -3c \). Then,

\[
|2000 - 3S| = e^{-0.03t+c_1} = e^{-0.03t}e^{c_1} = c_2e^{-0.03t},
\]

where \( c_2 = e^{c_1} \) (so \( c_2 \) must be a positive constant). Removing the absolute value, we have

\[
2000 - 3S = \pm c_2e^{-0.03t}.
\]
The choice of $c_2$ depends solely on whether or not $2000 - 3S$ is less than, equal to, or greater than zero, so we can again replace it with another constant $c_3$, and solving for $S$ we have

$$S = \frac{2000 - c_3e^{-0.03t}}{3} = \frac{2000}{3} + c_4e^{-0.03t}$$

where $c_4 = -c_3/3$.

**Slope Fields**

**Slope fields** are a way to visualize the solutions to a differential equation $\frac{dy}{dt} = f(t, y)$ without actually solving them. This is a very useful tool since many differential equations cannot be solved analytically anyhow! Creating a slope field is simply a matter of plugging points ($x$ and $y$ coordinate pairs) into the differential equation and drawing a little tangent "filament" with the resulting slope at each of those points. This is tedious and best done on a calculator or computer. However, a couple of special cases are worth considering.

**Slope fields for $\frac{dy}{dt} = f(t)$**

If the RHS is a function of $t$ only, then the derivatives only depend on $t$ as well. In other words, the slopes of the tangents will not change as $y$ changes, but only as $t$ changes. For example, the slope field along with some particular solutions for the differential equation $\frac{dy}{dt} = 2t$ are shown below.

Note that for any given $t$, the slopes of the tangents remain constant as one moves vertically through different $y$-values. This means that solutions are simply vertical translations of one another, as evidence by the solution to the differential equation, which will be of the form

$$y = \int f(t) \, dt + C,$$

where the $+C$ is the vertical shift.

**Slope fields for $\frac{dy}{dt} = f(y)$ (autonomous equations)**

Here the RHS depends only on $y$ and not on $t$, so tangents at the same $y$ coordinate will be equal as one moves from left to right. Analogous to the case discussed previously, solutions of autonomous equations are horizontal translations of each other. The slope field and some particular solutions for the autonomous
differential equation \( \frac{dy}{dt} = 4y(1 - y) \) are shown below.

Slope field for the mixture problem

Recall the mixture problem and its associated differential equation

\[
\frac{dS}{dt} = \frac{2000 - 3S}{100}
\]

having solution

\[
S(t) = ce^{-0.03t} + \frac{2000}{3}.
\]

The slope field along with a few particular solutions are shown below.

According to the original differential equation, when \( S < 2000/3 \), \( \frac{dS}{dt} > 0 \) and the solutions are increasing. When \( S = 2000/3 \), \( \frac{dS}{dt} = 0 \) and we have an equilibrium solution. Finally, when \( S > 2000/3 \), \( \frac{dS}{dt} < 0 \) and the solutions are decreasing. This is all confirmed on the slope field. In addition, we see that regardless of the starting amount of sugar in the vat, it appears that the amount of sugar will always head towards the equilibrium of \( S = 2000/3 \) as \( t \to \infty \).

Existence and Uniqueness of Solutions

Before we get too far into a discussion of the solution of differential equations, it would be nice to know whether or not a given equation actually has a solution. Luckily, we are guaranteed existence of a solution under specific conditions, namely:
**Theorem 14 (Existence)** Suppose \( \frac{dy}{dx} = f(x, y) \) is continuous in some rectangle 

\[ B = \{(x, y) | a < x < b, \ c < y < d\} \]

in the \( xy \)-plane. If \((x_0, y_0)\) is a point in this rectangle, then there exists an \( \epsilon > 0 \) and a function \( y(x) \) defined for \( x_0 - \epsilon < x < x_0 + \epsilon \) that solves the IVP \( \frac{dy}{dx} = f(x, y) \) with \( y(x_0) = y_0 \).

**Interpretation:** All we need to guarantee existence of a solution to an IVP, at least on some \( x \)-interval that is possibly smaller than the rectangle’s width, is for the derivative to be continuous for some region surrounding the initial value \((x_0, y_0)\). Let’s make some observations regarding this theorem:

- If the function on the RHS is "well-behaved," solutions will exist.
- The theorem does NOT tell us how to find the solution(s); it only tells us that there ARE solutions.
- The solution is only guaranteed to exist over an interval of \((x-\epsilon, x+\epsilon)\), and the only qualification of \( \epsilon \) is that it’s positive, so \( \epsilon \) could be very small, meaning the solution might only exist over a very small interval around the initial \( x \)-value \( x_0 \).
- The solution can still exist outside of the rectangle \( B = \{(x, y) | a < x < b, \ c < y < d\} \). We’re only **guaranteed** existence within that rectangle.

**Example 15** Consider the IVP 

\[ \frac{dy}{dt} = 1 + y^2, \quad y(0) = 0. \]

This is easily solved using analytic methods:

\[
\int \frac{dy}{1 + y^2} = \int dt \\
\tan^{-1} y = t + C \\
y = \tan (t + C)
\]

and with \( y(0) = 0 \),

\[ C = 0 \pm k\pi. \]

**Solutions to this equation "blow up" at all odd multiples of \( \pi/2 \). Therefore, the solutions do not exist at these points (because \( f(t, y) = 1 + y^2 \) is continuous, we are guaranteed a solution for a given initial condition, say \( y(0) = 0 \), however, it does not exist for all time \( t \)).**

When we talk about the existence of solutions, we often talk about uniqueness of solutions as well. How do we know that a solution is the solution? This is an important question as well - if solutions were not unique, we’d always have to worry about multiple solutions for every initial value problem, and different solutions through the same initial point could yield entirely different predictions for the same system (which would be a problem in physical and mechanical applications)! Thankfully we have a theorem to help address this question as well:

**Theorem 16 (Uniqueness)** Suppose in addition that \( \frac{\partial f}{\partial y} \) is also continuous in the rectangle 

\[ B = \{(x, y) | a < x < b, \ c < y < d\} \] .

Then the solution of the IVP is unique. In other words, if \( y_1(t) \) and \( y_2(t) \) are both solutions to the same IVP 

\[ \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \]

for all \( x \) for which the solution is guaranteed to exist (i.e., for \( x_0 - \epsilon < x < x_0 + \epsilon \)), then it must be true that \( y_1(x) \) and \( y_2(x) \) are the same solutions, i.e., \( y_1(x) = y_2(x) \).
Example 17 (Non-uniqueness) Consider the differential equation
\[ \frac{dy}{dt} = 3y^{2/3}. \]

The RHS is a continuous function on the entire $ty$-plane (so the hypothesis of the Existence Theorem is met). However,
\[ \frac{\partial f}{\partial y} = 2y^{-1/3} \]
does not exist if $y = 0$, so we are not allowed to use the Uniqueness Theorem.

Notice the last line in the above example. The partial derivative with respect to $y$ of $f(t, y)$ is not continuous. This does not necessarily mean that the solution is not unique. It simply means that we cannot use this particular theorem to come to any conclusion about uniqueness! Any given solution in this case may or may not be unique.

Example 18 (Continued) In this case, antidi\-erentiation yields
\[ \int y^{-2/3}dy = \int 3dt \]
\[ y = (t + c)^3. \]

Now, if the initial condition had been $y(0) = 0$, it is easy to see from the original differential equation that $y(t) = 0$ is an equilibrium solution. However, with $c = 0$ we also have $y(t) = t^3$ as a solution. We therefore have two different solutions to the same initial value problem, i.e., the solutions are not unique.

The implication of this theorem is that solution curves that meet these conditions cannot intersect each other, and this is a very important realization in the qualitative analysis of solutions. Consider the following examples.

Example 19 Given the initial value problem
\[ \frac{dy}{dt} = \frac{(y^2 - 4)(\sin^2 y^3 + \cos y - 2)}{2}, \quad y(0) = \frac{1}{2}, \]
it is easy to spot $y = 2$ as an equilibrium solution. However, although this is a separable equation, it would be rather difficult to determine the actual solution analytically. However, the uniqueness theorem tells us that because $y = 2$ is a solution, the solution with initial condition $y(0) = 1/2$ cannot cross or even touch the solution $y = 2$.

The uniqueness theorem is an important qualitative tool, as the examples above and below show.
Example 20 Consider the differential equation
\[ \frac{dy}{dt} = \frac{(1+t)^2}{(1+y)^2}. \]
It should be easy to convince yourself that \( y_1(t) = t \) is a solution to the IVP with initial condition \( y(0) = 0 \) because if \( y_1(t) = t \), then
\[ \frac{dy}{dt} = \frac{(1+t)^2}{(1+y)^2} = \frac{d}{dt} (t). \]
By uniqueness, if \( y_2(t) \) satisfies the initial condition \( y(0) = -0.1 \), then we know \( y_2(0) < y_1(0) \), so it must be true that \( y_2(t) < y_1(t) \) for all \( t \), and hence \( y_2(t) < t \) for all \( t \).

Example 21 Now consider the autonomous differential equation
\[ \frac{dy}{dt} = (y-2)(y+1). \]
Note that \( y = 2 \) and \( y = -1 \) are equilibrium solutions, so by the uniqueness theorem (check that the hypotheses are met!), we know that any solutions with initial values anywhere between \( y = -1 \) and \( y = 2 \) will stay between \( y = -1 \) and \( y = 2 \) for all \( t \). In this case, we have further information. Note that \( \frac{dy}{dt} > 0 \) for \( y < -1 \) and for \( y > 2 \), while \( \frac{dy}{dt} < 0 \) for \(-1 < y < 2 \). Hence solutions between the equilibrium solutions will decrease for all time, those below \( y = -1 \) will increase towards \( y = -1 \) as \( t \to \infty \), and those above \( y = 2 \) will increase away from \( y = 2 \) as \( t \to \infty \).

For further explanation, be sure to read and understand Examples 5 and 6 as well as remarks 1, 2, and 3 on pages 22-25 of your textbook. Keep the following in mind as you study Example 6. The same differential equation might have solutions exhibiting different behaviors, depending on the initial conditions. The equation \( x \frac{dy}{dx} = 2y \) has
- a unique solution near \((a, b)\) if \( a \neq 0 \);
- no solution if \( a = 0 \) but \( b \neq 0 \);
- infinitely many solutions if \( a = b = 0 \); and
- solutions that start as unique (and satisfy the above theorems) might branch off into many solutions outside of rectangle \( B \). In other words, a solution can still exist on an interval larger than one on which that solution is unique or guaranteed to be unique.

**Equilibrium and the Phase Line**

Our exploration of exponential and logistic population models serves as an excellent example and introductory precursor to this section. In our previous work, we looked for equilibrium solutions to differential equations. These equilibrium solutions are called constant solutions, because the line \( P(t) = C \), where \( C \) is a constant, represented those solutions (at least in the population examples). Recall too that the population model differential equations only involved \( P(t) \), and not \( t \) itself (for example, the logistic equation \( P'(t) = kP(M-P) \) does not have a \( t \) on the right-hand side of the equation). Equations with this characteristic are called autonomous differential equations. It should be clear that the slope of a constant solution is zero, hence for an autonomous equation, the constant solutions occur when \( \frac{dx}{dt} = f(x) = 0 \), and these solutions of \( f(x) = 0 \) are called critical points of the differential equation.

Without getting into technical details (see page 92 if you wish), we can consider an equilibrium solution stable (also called a sink) if solutions starting "near" that solution tend toward it as \( t \to \infty \), unstable (also called a source) if solutions starting "near" that solution tend away from it as \( t \to \infty \), and semistable (also called a node) if solutions starting slightly "above" are stable and those starting "below" are unstable (or vice versa). These classifications of solutions near equilibria should be very easy to spot on direction field plots (more on this later).
Direction Fields and Phase Diagrams

Another tool for the analysis of equilibria is called the **phase diagram**. This is simply a vertical line (typically - for some reason, our text uses a horizontal line) that has the zeros of the derivative indicated. Much like we did with "sign charts" in analyzing the graphs of $f$, $f'$, and $f''$ in calculus, we now consider the sign of the derivative between these "critical points". For example, if the derivative is negative above a particular equilibrium solution, we know that the solution is heading down toward that equilibrium.

As an example, consider a fish population modeled by the logistic equation $\frac{dP}{dt} = P(4-P)$, (where $M = 4$ is measured in hundreds of fish). Regardless of the initial population, the eventual population of fish would be 400. However, if we introduce a harvesting term (as done in the lab assignment) of $h = 3$, so that 300 fish are harvested annually at a presumably constant rate throughout the year, the differential equation then becomes

$$\frac{dP}{dt} = P(4-P) - 3.$$  (4)

If we seek the constant solutions to the right-hand side of Eq. (4), we are led to the quadratic equation $-P^2 + 4P - 3 = (3-P)(P-1) = 0$ with solutions $x = 1$ and $x = 3$. As shown in the direction field, we see that $P = 100$ has become the new "threshold" population and $P = 300$ has become the new "limiting" population.² Below is a direction field with a few solutions plotted, and afterwards is an example of a phase diagram:

![Direction Field and Phase Diagram](image)

Obviously, the direction field and the phase diagram are tools that are quite complementary of each other (and perhaps even complimentary ... we'll never know). Consider the following examples on your own, in which you should sketch phase lines and rough direction fields (i.e., show the direction of solutions in each region created by the equilibrium solutions)

1. $\frac{dy}{dt} = (y - 2) (y + 1)$
2. $\frac{dy}{dt} = (1 - y) y$
3. $\frac{dy}{dt} = (y - 2) (y + 3)$
4. $\frac{dy}{dt} = \sin y$
5. $\frac{dy}{dt} = y \cos y$
6. $\frac{dy}{dt} = (2 - y) \sin y$
7. $\frac{dP}{dt} = \left(1 - \frac{P}{20}\right)^3 \left(\frac{P}{5} - 1\right) P^7$

²Recall that the "threshold" population is the critical population at which initial populations below will die off to extinction while initial populations above will grow toward the limiting population. Meanwhile, the limiting population is what we've typically called the "carrying capacity."
Not all solutions exist for all time - Consider the following examples.

**Example 22** The equation

\[
\frac{dy}{dt} = (1 + y)^2
\]

has an equilibrium point at \( y = -1 \) and \( dy/dt > 0 \) for all \( y \neq 0 \). In addition, as \( y \) increases above \( y = -1 \), the derivative grows larger and larger, meaning the solution curves speed up as they move away from the equilibrium solution. Similarly, as solutions from below increase towards \( y = -1 \), the value of the derivative gets closer and closer to zero, so the solution curves "slow down" as they approach the equilibrium solution. By uniqueness, the solution approaches the equilibrium but never gets there, and so we say the solution approaches the equilibrium solution asymptotically. By the way, the analytic solution to this equation is given by

\[
y(t) = y(0) + \frac{t}{t + c}.
\]

For \( y(0) > -1 \), it must be true that the quantity \( y(t) = 1 - 1/t + c \) > -1, meaning \( c < 0 \). Hence the solutions are only defined for \( t < -c \), as as \( t \to -c \) from below, the solutions tend towards \( -\infty \).

**Example 23** The equation

\[
\frac{dy}{dt} = \frac{1}{1-y}
\]

implies that \( dy/dt < 0 \) if \( y > 1 \), \( dy/dt > 0 \) if \( y < 1 \) and \( dy/dt \) does not exist when \( y = 1 \). The phase line has a "gap" or a "hole" in it!

Solutions all approach \( y = 1 \) as \( t \to \infty \), and since \( dy/dt \) gets larger and larger as \( y \to 1 \), the solutions speed up as they approach \( y = 1 \) and they reach the "hole" there in finite time.

**Bifurcations**

The behavior of the solutions to the above logistic equation (Eq. (4)) depends entirely on the harvesting term \( h \). Note that the equation

\[
\frac{dP}{dt} = P(4 - P) - h
\]

has critical points

\[
P(4 - P) - h = 0
\]

\[
P^2 - 4P + h = 0
\]

\[
P = 2 \pm \sqrt{4 - h},
\]

(5)
where \(2 + \sqrt{4-h}\) is the "carrying capacity" and \(2 - \sqrt{4-h}\) is the "threshold" population. As in any analysis of the roots of a quadratic, we have three possibilities, namely \(4-h > 0\) (two real roots), \(4-h = 0\) (one double root), or \(4-h < 0\) (complex roots). These three cases in turn correspond to \(h < 4\), \(h = 4\), and \(h > 4\), respectively.

With \(h < 4\) (as it was in our example above), we end up with two distinct equilibria, as shown in the direction field above. If \(h = 4\), then we have a single equilibrium solution at \(P = 2\), and its direction field is shown in figure 2.2.10 on page 95. Note that with this particular harvesting rate, populations that start above 200 eventually stabilize at 200, while those that start below 200 die off to extinction! Finally, if \(h > 4\), we end up with no real solutions, hence no equilibria, and so regardless of the initial population, the fish will die off to extinction (see the direction field in figure 2.2.11 on page 96). In conclusion, we have

- two critical points if \(h < 4\);
- one critical point if \(h = 4\);
- no critical point if \(h > 4\).

For this reason, we call \(h = 4\) a bifurcation point for the differential equation with parameter \(h\). We can often plot the bifurcations via a bifurcation diagram whereby the critical points \(c\) are related to the bifurcation parameter (\(h\) in this case). The critical points represent population values, so if we simply rename Eq. (5) as

\[
c = 2 \pm \sqrt{4-h},
\]

we can plot the curve as \((c - 2)^2 = 4 - h\), which of course is a parabola with vertex \((4, 2)\) that opens to the left (considering \(c\) as the dependent variable and \(h\) as the dependent variable):

Then, for each value of \(h\) we can see where the critical points (i.e., the equilibria) of the differential equation in Eq. (4) will be.

**Linear Differential Equations**

We know how to solve separable differential equations provided the resulting integrals are integrable. However, most differential equations are actually not separable. There is no single general technique that allows us to solve any differential equation, so we need to develop and use methods that are specific to particular types of equations. We will learn a small subset of the possible solution techniques throughout this course, and the first such technique is for the solution of what are known as linear first-order ordinary differential equations. A first-order equation is linear if it can be written as

\[
\frac{dy}{dt} = a(t) y + b(t),
\]

where \(a(t)\) and \(b(t)\) are each arbitrary functions of \(t\). The equation is linear because the dependent variable, \(y\), only appears to the first power. The equation is ordinary because it does not involve partial derivatives.
And finally, the equation is first-order because the first derivative is the highest-order derivative present in the equation. Some examples of linear equations are

\[
\frac{dy}{dt} = t^2 y + \cos t
\]
\[
\frac{dy}{dt} = e^{4 \sin t / t^3 + 7t} y + 23t^3 - 7t^2 + 3
\]

with \(a(t) = t^2\) and \(b(t) = \cos t\) in the first case and \(a(t) = e^{4 \sin t / t^3 + 7t}\) and \(b(t) = 23t^3 - 7t^2 + 3\) in the second case. Also,

\[
\frac{dy}{dt} - 3y = ty + 2
\]

is linear because it can be rewritten as

\[
\frac{dy}{dt} = (3 + t) y + 2.
\]

Furthermore, the equation

\[
\frac{dy}{dt} = 2y + 8
\]

is not only linear, but it is autonomous and hence separable. However,

\[
\frac{dy}{dt} = t \sin y
\]

is not linear, but is separable.

**Homogeneous vs. nonhomogeneous equations** - If there is no \(b(t)\) in the equation (i.e., \(b(t) = 0\) for all \(t\)), the resulting equation, which is of the form

\[
\frac{dy}{dt} = a(t) y
\]

is called homogeneous or unforced. Otherwise, the equation

\[
\frac{dy}{dt} = a(t) y + b(t)
\]

is called nonhomogeneous or forced. Additionally, if the equation is such that \(a(t) = \lambda\), where \(\lambda\) is a constant, the equation

\[
\frac{dy}{dt} = \lambda y + b(t)
\]

is a constant-coefficient equation.

As it turns out, linear differential equations are very important. We use them extensively in modeling problems (think of radioactive growth, Newton’s law of cooling, mixing, etc.). Because of this fact the modeling process usually starts with an attempt at a linear model. Probably the most important aspect of linear equations is that there is a nice relationship between their solutions. There are two principles of linearity that we will make regular use of throughout the course, one for homogeneous equations and the other for nonhomogeneous equations.

**Linearity principle for homogeneous equations:** If \(y_h(t)\) is a solution of the homogeneous linear equation

\[
\frac{dy}{dt} = a(t) y,
\]

then any constant multiple of \(y_h(t)\) is also a solution. That is to say, \(k y_h(t)\) is a solution for any constant \(k\). (Verify this!) As an aside, note that the equilibrium solution \(y(t) = 0\) is a solution to every homogeneous linear equation. As a quick generalization, note that in the homogeneous case, we have a separable equation, and

\[
\int \frac{dy}{y} = \int a(t) \, dt
\]
\[
\ln |y| + c = \int a(t) \, dt
\]
\[
y(t) = k e^{\int a(t) \, dt},
\]
by which it is easy to see that all solutions are constant multiples of $e^{\int a(t)dt}$. For example, all solutions to

$$\frac{dy}{dt} = (\cos t) y$$

would be multiples of

$$y(t) = e^{\int \cos t dt} = e^{\sin t}.$$

**Linearity principle for nonhomogeneous equations:** Suppose we have a nonhomogeneous equation

$$\frac{dy}{dt} = a(t) y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t) y.$$

1. If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ is any solution of the nonhomogeneous equation, then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $y_h(t)$ is nonzero, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation. In other words, we can say that the general solution of the nonhomogeneous equation is the sum of the general solution of the homogeneous equation and one solution of the nonhomogeneous equation. For example, we know that the general solution to the homogeneous equation

$$\frac{dy}{dt} = (\cos t) y$$

is given by

$$y(t) = ke^{\sin t}.$$ 

If we had the nonhomogeneous equation

$$\frac{dy}{dt} = (\cos t) y + \frac{1}{5} (1 - t \cos t)$$

for which we happen to know $y_p(t) = t/5$ is a solution, our overall general solution would be

$$y(t) = ke^{\sin t} + t/5.$$ 

The above reasoning leads us to a three-step process for solving a linear equation:

1. Find a general solution for the homogeneous equation.

2. Find one "particular" solution for the nonhomogeneous equation.

3. Add the solutions found in parts (1) and (2) to obtain the general solution to the nonhomogeneous equation.

Not all is good news, though, as finding a particular solution in step 2 can be quite problematic. Our best techniques for when the linear equations are not constant coefficient equations require a refined guessing technique shown in the following examples.
Example 24  Consider the nonhomogeneous linear differential equation
\[
\frac{dy}{dt} = -2y + e^t
\]
with associated homogeneous equation \( \frac{dy}{dt} = -2y \). By now we know that the solution to this homogeneous equation is given by \( y(t) = ke^{-2t} \) (obtainable by separation of variables if necessary). As for the particular solution to the nonhomogeneous equation, we rewrite the equation with all terms involving \( y \) on the left:
\[
\frac{dy}{dt} + 2y = e^t.
\]
We need a function \( y_p(t) \) that when inserted into the LHS yields the RHS. We probably shouldn’t use trigonometric functions or polynomials, as none of these are present on the right. However, we know that exponentials are their own derivatives. A guess such as \( y_p(t) = e^t \) seems to make sense. If we try this, we get
\[
\frac{d}{dt} (e^t) + 2(e^t) = 3e^t,
\]
which of course is not what we had on the RHS, so this is not the correct guess. But it is so very close.

The following method, called the "Method of Undetermined Coefficients," is a method by which we can streamline this guessing process by letting the equation help us determine the desired function. Let’s see what happens if we guess \( Ae^t \) instead of just \( e^t \):

Example 25  (Continued) With \( y_p(t) = Ae^t \) we have
\[
\frac{d}{dt} (Ae^t) + 2(Ae^t) = 3Ae^t,
\]
and since the RHS is supposed to be \( e^t \), we have
\[
3Ae^t = e^t \Rightarrow A = \frac{1}{3} \text{ or } y_p(t) = \frac{1}{3}e^t,
\]
and the general solution to the nonhomogeneous equation is this solution added to the general solution of the associated homogeneous equation, or
\[
y(t) = \frac{1}{3}e^t + ke^{-2t},
\]
where \( k \) is an arbitrary constant.

Example 26  Now consider
\[
\frac{dy}{dt} + 2y = \cos 3t.
\]
The general solution for the associated homogeneous equation is still \( y(t) = ke^{-2t} \). This time, though, an exponential guess for \( y_p(t) \) doesn’t make sense. A trigonometric guess does make sense, but if we only guess \( y_p(t) = A \cos 3t \), our LHS would contain sines and cosines whereas the RHS does not. Instead, we make the guess
\[
y_p(t) = A \cos 3t + B \sin 3t.
\]
Upon substitution into the original differential equation, we have
\[
\frac{d}{dt} (A \cos 3t + B \sin 3t) + 2 (A \cos 3t + B \sin 3t) = -3A \sin 3t + 3B \cos 3t + 2A \cos 3t + 2B \sin 3t
\]
\[
= (-3A + 2B) \sin 3t + (3B + 2A) \cos 3t.
\]
In order for this last result to be a solution, it must be true that
\[
\cos 3t = (-3A + 2B) \sin 3t + (3B + 2A) \cos 3t,
\]
and we see the system of equations
\[
\begin{align*}
-3A + 2B &= 0 \\
2A + 3B &= 1
\end{align*}
\]
having \( A = 2/13 \) and \( B = 3/13 \) as solutions. Therefore we can conclude that

\[
y_p(t) = \frac{2}{13} \cos 3t + \frac{3}{13} \sin 3t
\]

and the general solution for the original equation is

\[
y(t) = ke^{-2t} + \frac{2}{13} \cos 3t + \frac{3}{13} \sin 3t.
\]

The above example is interesting in its own qualitative way. Graph a number of solutions for different values of \( k \). What appears to happen to all solution curves as time progresses? Note that the general solution of the associated homogeneous equation tends to zero quickly, while the solution of the nonhomogeneous equation serves as a \textbf{steady-state solution} (every solution tends toward it in the long term).

**Example 27** Adjust the equation one more time to arrive at

\[
\frac{dy}{dt} = -2y + 3e^{-2t},
\]

having the same general solution to the associated homogeneous equation \( y_h(t) = ke^{-2t} \). This time, it makes sense to try \( y_p(t) = Ae^{-2t} \). Then

\[
\frac{d}{dt} (Ae^{-2t}) + 2 (Ae^{-2t}) = -2Ae^{-2t} + 2Ae^{-2t} = 0,
\]

which obviously cannot equal the RHS \( 3e^{-2t} \) for any value of \( A \). What has gone wrong? In this case, our guess "matches" the form of the homogeneous solution. Since our guess was actually a solution to the homogeneous equation, the LHS will always give us zero! So, to adjust this, we need to make a second guess that contains \( e^{-2t} \), is not a solution of the homogeneous equation, and is as simple as possible. One such guess (more on this later) is

\[
y_p(t) = Ate^{-2t}.
\]

Upon substitution into the original equation, we obtain

\[
\frac{d}{dt} (Ate^{-2t}) + 2 (Ate^{-2t}) = Ae^{-2t} - 2Ate^{-2t} + 2ate^{-2t} = Ae^{-2t}.
\]

Since we need this to be equivalent to what was in the original equation, namely \( 3e^{-2t} \), we must have \( A = 3 \), and our particular solution must be \( y_p(t) = 3te^{-2t} \). Finally, the resulting general solution to the original equation would be

\[
y(t) = ke^{-2t} + 3te^{-2t},
\]

where \( k \) is an arbitrary constant.

**Observation** For any first-order linear nonhomogeneous equation of the form

\[
\frac{dy}{dt} = ky + g(t)
\]

the associated homogeneous equation will be

\[
\frac{dy}{dt} = ky
\]

having general solution \( y = Ce^{kt} \). If \( k < 0 \), this general solution will tend toward zero, and so all solutions eventually will tend toward the solution \( y_p(t) \). On the other hand, if \( k \geq 0 \), differing behaviors are possible. For example, compare

\[
\frac{dy}{dt} + 2y = g(t) \quad \text{to} \quad \frac{dy}{dt} - 2y = g(t)
\]

for \(-1 < g(t) < 2 \) for all \( t \).
Linear Equations and Integrating Factors

Another method for solving first-order (but only first-order) linear differential equations involves using what is called an integrating factor. Consider a linear equation with constant coefficients

\[ y' + ay = f(t). \]  

(6)

If we multiply both sides of this equation by \( e^{at} \), a technique first employed by Euler (who else?!?), we have

\[ e^{at} (y' + ay) = f(t) e^{at}. \]  

(7)

But wait! Isn’t the LHS above simply the result of taking the derivative of the product \( e^{at}y \)?

\[ \frac{d}{dt} (e^{at}y) = e^{at}y' + ae^{at}y = e^{at} (y' + ay) \]  

(8)

Sho iz, bro!! So, combining (7) and (8), we have

\[ \frac{d}{dt} (e^{at}y) = f(t) e^{at} \]

We can then integrate both sides of (7), obtaining

\[ e^{at}y = \int f(t) e^{at} dt + C. \]

Then, solving for \( y \), we find

\[ y(t) = e^{-at} \int f(t) e^{at} dt + Ce^{-at}. \]

A similar approach works in the case of variable coefficients, i.e., for equations of the form

\[ \frac{dy}{dt} + a(t)y = b(t), \]

(9)

where \( a(t) \) is now a function of \( t \) instead of a constant \( a \). We "simply" need to invent the function \( \rho(t) \) (we used \( e^{at} \) above, in (7)) so that when we multiply both sides of (9) by this function, the LHS is the derivative of the product \( \rho(t) \) and \( y(t) \):

\[ \rho(t) \frac{dy}{dt} + \rho(t) a(t) y(t) = \rho(t) b(t) \]

and

\[ \frac{d}{dt} [\rho(t) y(t)] = \rho(t) \frac{dy}{dt} + \rho(t) a(t) y(t). \]  

(10)

How can we find such a function \( \rho(t) \)? Work out the derivative on the LHS of (10), yielding

\[ \frac{d}{dt} [\rho(t) y(t)] = \rho(t) \frac{dy}{dt} + y(t) \frac{d\rho}{dt} + \rho(t) \frac{dy}{dt} \]

then, equating this result with that from the previous equation, we see that

\[ \rho(t) \frac{dy}{dt} + \rho(t) a(t) y(t) = \rho(t) \frac{dy}{dt} + y(t) \frac{d\rho}{dt} \]

\[ \rho(t) a(t) y(t) = y(t) \frac{d\rho}{dt} \]

and hence

\[ \rho(t) a(t) = \frac{d\rho}{dt} \]

whereby we note that this is simply a linear homogeneous first-order and hence separable equation

\[ \frac{d\rho}{\rho(t)} = a(t) dt \]

\[ \ln |\rho(t)| = \int a(t) dt + C_1 \]

\[ \rho(t) = Ce^{\int a(t) dt}. \]  

(11)
So, it appears that we can indeed find such a function \( \rho(t) \), provided we can actually carry out the required integration in (11). Furthermore, it really does not matter which integrating factor we use, so we generally select \( C \) to be the most convenient (usually \( C = 1 \), or \( C_1 = 0 \)).

**Example 28** Consider the nonhomogeneous equation \( x^2 \frac{dy}{dx} + 2xy = x + 3 \).

Here, we first rewrite the equation in the form \( y' + a(x)y = b(x) \):

\[
\frac{dy}{dx} + \frac{2}{x}y = \frac{x + 3}{x^2},
\]

from which we see that an integrating factor will be

\[
\rho(x) = e^{\int \frac{2}{x} \, dx} = e^{\ln(x)} = x^2.
\]

Multiplying through by \( \rho(x) = x^2 \) gives

\[
\frac{dy}{dx}x^2 + 2xy = x + 3
\]

in which we recognize the LHS as the derivative of the product \( \frac{d}{dx} [x^2y] \):

\[
\frac{dy}{dx}x^2 + 2xy = \frac{d}{dx} [x^2y] = x + 3
\]

or more concisely,

\[
\frac{d}{dx} [x^2y] = x + 3.
\]

Now, integrating both sides with respect to \( x \), we find

\[
\int \frac{d}{dx} [x^2y] \, dx = \int (x + 3) \, dx
\]

\[
x^2y = \frac{x^2}{2} + 3x + C
\]

\[
y = \frac{1}{2} + \frac{3}{x} + \frac{C}{x^2}.
\]

**Example 29** Find a general solution of the equation \( (x^2 + 1) \frac{dy}{dx} + 3xy = 6x \).

Again, rewrite the equation as

\[
\frac{dy}{dx} + \frac{3xy}{x^2 + 1} = \frac{6x}{x^2 + 1}.
\]

whence we find the integrating factor

\[
\rho(x) = e^{\int \frac{3x}{x^2 + 1} \, dx} = e^{\frac{3}{2} \int \frac{du}{u^2}} = e^{\ln(x^2 + 1)^{3/2}} = (x^2 + 1)^{3/2}.
\]

Multiplying the equation through by \( \rho(x) \) produces

\[
(x^2 + 1)^{3/2} \frac{dy}{dx} + (x^2 + 1)^{3/2} \frac{3xy}{x^2 + 1} = (x^2 + 1)^{3/2} \frac{6x}{x^2 + 1}
\]

\[
\frac{d}{dx} [ (x^2 + 1)^{3/2} y ] = 6x (x^2 + 1)^{1/2}.
\]

Integrating both sides with respect to \( x \) gives us

\[
(x^2 + 1)^{3/2} y = \int 3u^{1/2} \, du
\]

\[
(x^2 + 1)^{3/2} y = 2 (x^2 + 1)^{3/2} + C
\]

and finally,

\[
y = 2 + \frac{C}{(x^2 + 1)^{3/2}}.
\]
Observation Note in both of the examples above that the integrating factor method produces both a general solution to the associated homogeneous equation \((y_h)\) as well as a particular solution to the given nonhomogeneous equation \((y_p)\), but simultaneously! Specifically, for the first example, we have

\[
y = \frac{1}{2} + \frac{3}{x} + \frac{C}{x^2}.
\]

Verification of these two solutions is straightforward. If \(y_h = Cx^{-2}\), then \(y_h' = -2C x^{-3}\) and the LHS of the original differential equation in (12) becomes

\[
\frac{dy}{dx} + \frac{2}{x} y = -2Cx^{-3} + \frac{2}{x} = 0.
\]

Similarly, if \(y_p = \frac{1}{2} + \frac{3}{x}\), then \(y_p' = -3/x^2\), and the LHS of the original equation becomes

\[
\frac{dy}{dx} + \frac{2}{x} y = -3\frac{1}{x^2} + \frac{2}{x} \left(\frac{1}{2} + \frac{3}{x}\right) = \frac{x + 3}{x^2},
\]

which is the RHS of the original equation. You should confirm the results of the second example as well, where we have

\[
y = \frac{2}{y_p} + \frac{C}{(x^2 + 1)^{3/2}}.
\]

A Mixture Problem Revisited

Now that we have addressed first-order linear differential equations, we can attack more complicated mixture problems. Let’s revisit our earlier example, where we had a large vat of sugar water for the making of soft drinks. We’ll make one slight change to the scenario, as indicated in boldface type below:

- The vat at first contains 100 gallons of liquid.
- The vat is uniformly mixed so the concentration of sugar is the same throughout.
- Sugar water containing 5 tbsp. of sugar per gallon enters the vat through pipe A at a rate of 2 gal/min.
- Sugar water containing 10 tbsp. of sugar per gallon enters the vat through pipe B at the rate of 1 gal/min.
- **Sugar water leaves the vat through pipe C at the rate of 4 gal/min.** (In our previous example, it left at 3 gal/min, the same rate at which it entered.)

The independent variable in this case is time \(t\). We could model in terms of the amount of sugar \(S(t)\) or in terms of the concentration \(C(t)\). We’ll use \(S(t)\). The rate of change in the amount of sugar in the vat is the difference between the amount of sugar being added and the amount of sugar being removed. The model (with units) is

\[
\frac{dS}{dt} \text{ tbsp/min} = \frac{2 \text{ gal/min} \cdot 5 \text{ tbsp/gal}}{\text{input from pipe A (tbsp/min)}} + \frac{1 \text{ gal/min} \cdot 10 \text{ tbsp/gal}}{\text{input from pipe B (tbsp/min)}} - \frac{4 \text{ gal/min}}{\text{output rate}} \cdot \frac{S(t) \text{ tbsp}}{\text{total concentration in the vat}} - \frac{S(t) \text{ tbsp}}{\text{output through pipe C (tbsp/min)}}
\]

Note the new denominator in the last term. The amount of sugar leaving the tank depends on the current concentration of sugar in the tank, which would be \(S\) total tablespoons per \(V\) gallons. However, now that the solution is leaving faster than it is coming in (leaving at 4 gal/min and coming in at a total of 3 gal/min), the number of gallons in the tank decreases by 1 gallon every minute. Therefore, the amount of solution in
the tank is the initial 100 gallons less 1 gallon per minute, or $100 - t$. So the concentration of the sugar in the tank at time $t$ is $\frac{S(t)}{100 - t}$. Our new differential equation is therefore given by

$$\frac{dS}{dt} = 20 - \frac{4S(t)}{100 - t},$$

or written in a now more recognizable form,

$$\frac{dS}{dt} + \frac{4}{100 - t}S(t) = 20.$$

This of course is a linear equation in $S(t)$. An appropriate integrating factor would be

$$e^{\int \frac{4}{100 - t} dt} = e^{-4 \ln(100 - t)} = (100 - t)^{-4}.$$

(Note - once $t \to 100$, the tank will be empty, so we need not concern ourselves with absolute value here.) Multiplying the equation by this integrating factor yields

$$(100 - t)^{-4} \frac{dS}{dt} + \frac{4}{100 - t}S(t) = 20 (100 - t)^{-4}$$

or

$$\frac{d}{dt} \left( (100 - t)^{-4} S(t) \right) = 20 (100 - t)^{-4}$$

$$(100 - t)^{-4} S(t) = \int 20 (100 - t)^{-4} dt$$

$$(100 - t)^{-4} S(t) = \frac{20}{3} (100 - t)^{-3} + C$$

$$S(t) = \frac{20}{3} (100 - t) + C (100 - t)^4.$$

Now, if we started with pure water, $S(0) = 0$, and so

$$0 = \frac{2000}{3} + 10^4 C \Rightarrow C = -\frac{20}{3 (100^3)},$$

and

$$S(t) = \frac{20}{3} (100 - t) - \frac{2}{300000} (100 - t)^4$$

$$= \frac{2000000 (100 - t) - 2 (100 - t)^4}{300000}$$

$$= 20t - \frac{2}{5} t^2 + \frac{1}{375} t^3 - \frac{1}{150000} t^4.$$
We see now that the solution is a quartic, and it has roots when $t = 0, t = 100$. A graph of $S(t)$ shows that it reaches a maximum at approximately $t = 40$. 

![Graph of S(t)](image-url)